ARCHIMEDEAN L-FACTORS FOR STANDARD L-FUNCTIONS ATTACHED TO NON-HOLOMORPHIC SIEGEL MODULAR FORMS OF DEGREE 2 (Automorphic Representations and Related Topics)

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第 1871 巻 2013 年 84-89

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INTRODUCTION

Bump, Friedberg and Ginzburg [1] introduced a zeta integral which contains two complex variables s_1, s_2 and interpolates the standard and the spinor L-functions for generic cuspidal representation $\pi = \otimes_v' \pi_v$ of $\mathrm{GSp}(2, \mathbf{A})$. Actually they carried out the unramified computation to show that the local zeta integral coincides with the product $L(s_1, \pi_v, \mathrm{std})L(s_2, \pi_v, \mathrm{spin})$ of the local L-factors at the unramified place v. We compute the (real) archimedean zeta integral by using the explicit formulas of the Whittaker functions on $\mathrm{GSp}(2, \mathbf{R})$ developed by Oda, Miyazaki, Moriyama and the author. When π_∞ is isomorphic to a large discrete series representations, for an appropriate choice of Whittaker function and sections for Eisenstein series, we show that the archimedean zeta integral is equal to the product of two archimedean L-factors.

1. ZETA INTEGRALS

We recall the zeta integral discovered by Bump, Friedberg and Ginzburg [1]. Let G be the symplectic group with similitude of degree two defined over $\mathbf{Q}$:

$$\mathsf{G} = \mathrm{GSp}(2) = \{g \in \mathrm{GL}(4) \mid {}^t gJg = \nu(g)J \text{ for some } \nu(g) \in \mathrm{GL}(1)\}, \quad J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}.$$

Let $\mathbf{A}$ be the ring of adeles of $\mathbf{Q}$. Let $\pi = \otimes_v' \pi_v$ be a cuspidal automorphic representation of $\mathsf{G}(\mathbf{A})$. For simplicity, we assume that the central character of π is trivial. We take a maximal unipotent subgroup N_0 of G by

$$\mathsf{N}_0 = \{n(x_0, x_1, x_2, x_3) = \left(\begin{array}{cc|cc} 1 & x_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -x_0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} 1 & & x_1 & x_2 \\ & 1 & x_2 & x_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \in \mathsf{G}\}.$$

We fix a nontrivial additive character $\psi = \prod_v \psi_v : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^{(1)}$ and define a non-degenerate unitary character ψ_{N_0} of $\mathsf{N}_0(\mathbf{A})$ by $\psi_{\mathsf{N}_0}(n(x_0, x_1, x_2, x_3)) = \psi(-x_0 - x_3)$. For a cusp form $\varphi \in \pi$, the global Whittaker function W_φ attached to φ is defined by

$$W_\varphi(g) = \int_{\mathsf{N}_0(\mathbf{Q})\backslash \mathsf{N}_0(\mathbf{A})} \varphi(ng)\psi_{N_0}(n^{-1})\, dn, \quad g \in \mathsf{G}(\mathbf{A}).$$

Thoughout this paper we assume that π is generic, that is, W_φ does not vanish for some φ in π.

Let P_1 and P_2 be the Siegel and Jacobi (Klingen) parabolic subgroups of G, respectively:

$$\mathsf{P}_1 = \{\begin{pmatrix} * & * \\ 0_2 & * \end{pmatrix} \in \mathsf{G}\}, \quad \mathsf{P}_2 = \{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in \mathsf{G}\}.$$

The unipotent radical N_i of P_i is given by

$$\mathsf{N}_1 = \{n(0, x_1, x_2, x_3) \in \mathsf{G}\}, \quad \mathsf{N}_2 = \{n(x_0, x_1, 0, x_3) \in \mathsf{G}\}.$$

The Levi part of P_i is isomorphic to $\mathrm{GL}(2) \times \mathrm{GL}(1)$ embedded via the maps ι_i:

$$\iota_1(\alpha, g) = \begin{pmatrix} \alpha g & \\ & {}^t g^{-1} \end{pmatrix}, \quad \iota_2(\alpha, g) = \begin{pmatrix} \alpha & & & \\ & a & & b \\ & & \alpha^{-1}\det g & \\ & c & & d \end{pmatrix},$$

where $\alpha \in \mathrm{GL}(1)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$. The modulus characters δ_i of P_i are given by

$$\delta_1(\iota_1(\alpha, g)) = |\det g|^3 |\alpha|^3, \quad \delta_2(\iota_2(\alpha, g)) = |\det g|^{-2} |\alpha|^4.$$

For a complex number s, we denote by $\mathrm{Ind}_{\mathsf{P}_i(\mathbf{A})}^{\mathsf{G}(\mathbf{A})}(\delta_i^s)$ the space of smooth functions $f_i(s, g)$ on $\mathsf{G}(\mathbf{A})$ satisfying $f_i(s, pg) = \delta_i^s(p) f_i(s, g)$ for all $p \in \mathsf{P}_i(\mathbf{A})$ and $g \in \mathsf{G}(\mathbf{A})$. For complex numbers s_1 and s_2, we take a global sections $f_1 \in \mathrm{Ind}_{\mathsf{P}_1(\mathbf{A})}^{\mathsf{G}(\mathbf{A})}(\delta_1^{(s_1+1)/3})$ and $f_2 \in \mathrm{Ind}_{\mathsf{P}_2(\mathbf{A})}^{\mathsf{G}(\mathbf{A})}(\delta_2^{s_2/2+1/4})$. We define Eisenstein series $E_i(s_i, f_i, g)$ as usual manner:

$$E_i(s_i, f_i, g) = \sum_{\gamma \in \mathsf{P}_i(\mathbf{Q}) \backslash \mathsf{G}(\mathbf{Q})} f_i(s_i, \gamma g).$$

For a generic cusp form $\varphi \in \pi$, the global zeta integral is defined by

$$Z(s_1, s_2, \varphi, f_1, f_2) = \int_{\mathsf{Z}(\mathbf{A})\mathsf{G}(\mathbf{Q}) \backslash \mathsf{G}(\mathbf{A})} \varphi(g) E_1(s_1, f_1, g) E_2(s_2, f_2, g)\, dg.$$

Here we denote by Z the center of G. Unfolding two Eisenstein series, one can find the basic identity:

$$Z(s_1, s_2, \varphi, f_1, f_2) = \int_{\mathsf{Z}(\mathbf{A})\mathsf{N}_{12}(\mathbf{A}) \backslash \mathsf{G}(\mathbf{A})} W_\varphi(g) f_1(s_1, w_2 g) f_2(s_2, w_1 g)\, dg$$

for $\mathrm{Re}(s_1)$ and $\mathrm{Re}(s_2)$ sufficiently large. Here $\mathsf{N}_{12} = \mathsf{N}_1 \cap \mathsf{N}_2 = \{n(0, x_1, x_2, 0) \in \mathsf{G}\}$,

$$w_1 = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right) \text{ and } w_2 = \left(\begin{array}{cc|cc} 1 & & & \\ & & & -1 \\ \hline & & 1 & \\ & 1 & & \end{array}\right).$$

Suppose that π, f_1 and f_2 are factorizable. Then the global zeta integral is the product of local zeta integrals

$$Z_v(s_1, s_2, W_v, f_{1,v}, f_{2,v}) = \int_{Z(\mathbf{Q}_v)\mathsf{N}_{12}(\mathbf{Q}_v) \backslash \mathsf{G}(\mathbf{Q}_v)} W_v(g) f_{1,v}(s_1, w_2 g) f_{2,v}(s_2, w_1 g)\, dg,$$

where the subscripts denote the local analogues. The unramified computation is the following:

Theorem 1.1. [1, Theorem 1.2] *For $v = p < \infty$, we suppose that π_v is an unramified principal series representation of $\mathsf{G}(\mathbf{Q}_v)$. Let $\mathrm{diag}(\alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0\alpha_1\alpha_2) \in \mathrm{GSp}(2, \mathbf{C})$ be the Satake parameter of π_v. If all data are unramified, then we have*

$$Z_v(s_1, s_2, W_v, f_{1,v}, f_{2,v}) = \frac{L(s_1, \pi_v, \mathrm{std})L(s_2, \pi_v, \mathrm{spin})}{\{(1-p^{-(s_1+1)})(1-p^{-2s_1})(1-p^{-2s_2})\}^{-1}},$$

where the local L-factors are given by

$$L(s, \pi_v, \mathrm{std}) = \{(1-p^{-s})(1-\alpha_1 p^{-s})(1-\alpha_2 p^{-s})(1-\alpha_1^{-1}p^{-s})(1-\alpha_2^{-1}p^{-s})\}^{-1},$$
$$L(s, \pi_v, \mathrm{spin}) = \{(1-\alpha_0 p^{-s})(1-\alpha_0\alpha_1 p^{-s})(1-\alpha_0\alpha_2 p^{-s})(1-\alpha_0\alpha_1\alpha_2 p^{-s})\}^{-1}.$$

2. Representation theory of $\mathrm{GSp}(2, \mathbf{R})$

We introduce some representations of $G := \mathrm{GSp}(2, \mathbf{R})$. Main references are [6] and [7]. We denote by $G_0 = \mathrm{Sp}(2, \mathbf{R})$, $P_i = \mathsf{P}_i(\mathbf{R})$, $N_i = \mathsf{N}_i(\mathbf{R})$. Since we have assumed $\pi = \otimes'_v \pi$ is generic, each local component π_v is also generic. In particular, by a theorem of Kostant [2], the representation π_∞ of G must be large in the sense of Vogan [9]. An irreducible large representation π_∞ of G is equivalent to one of the following:

(i) a (limit of) large discrete series representation of G;
(ii) an irreducible (generalized) principal series representation induced from the parabolic subgroup P_i. ($i = 0, 1, 2$, P_0: Borel)

We mainly treat the case (i).

Fix a maximal compact subgroup K of G by $K = G \cap O(4)$. Let $K_0 = G_0 \cap O(4)$. Then K_0 is isomorphic to the unitary group $U(2)$ via the isomorphism $u : K_0 \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2)$. For $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2$ with $\lambda_1 \geq \lambda_2$, let $V_\lambda = \{f \in \mathbf{C}[x_1, x_2] \mid \text{homogeneous}, \deg(f) = \lambda_1 - \lambda_2\}$. For $f \in V_\lambda$ and $k \in K_0$, we set $(\tau_\lambda(k)f)(x_1, x_2) = (\det u(k))^{\lambda_2} f((x_1, x_2) \cdot u(k))$. Here $(x_1, x_2) \cdot u(k)$ means the ordinal product of matrices. Then $(\tau_\lambda, V_\lambda)$ is an irreducible $(\lambda_1 - \lambda_2 + 1)$-dimensional representation of K_0 with highest weight λ. We take a basis of V_λ as s $\{v_l \equiv v_l^\lambda := x_1^l x_2^{\lambda_1-\lambda_2-l} \mid 0 \leq l \leq \lambda_1 - \lambda_2\}$. A K_0-invariant inner product $\langle\ ,\ \rangle$ on V_λ is given by $\langle v_i, v_j \rangle = \delta_{i,j} \cdot \binom{\lambda_1 - \lambda_2}{i}^{-1}$.

For $(\lambda_1, \lambda_2) \in \mathbf{Z}^2$, we denote by $D_{(\lambda_1,\lambda_2)}$ be the (limit of) large discrete series representation of G_0 with Blattner parameter (λ_1, λ_2). Since $D_{(\lambda_1,\lambda_2)}$ is large, (λ_1, λ_2) satisfies $1 - \lambda_1 \leq \lambda_2 \leq 0$ or $1 + \lambda_2 \leq -\lambda_1 \leq 0$. For $c \in \mathbf{C}$, we denote by $D_{(\lambda_1,\lambda_2)}[c]$ the irreducible admissible representation of G characterized by $D_{(\lambda_1,\lambda_2)}[c]|_{G_0} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)}$ and $D_{(\lambda_1,\lambda_2)}[c](z) = z^c$ ($z > 0$).

Let $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. We fix $\psi_\infty(x) = \exp(2\pi\sqrt{-1}x)$ ($x \in \mathbf{R}$). When $\pi_\infty \cong D_{(\lambda_1,\lambda_2)}[c]$ with $1 - \lambda_1 \leq \lambda_2 \leq 0$, the L- and ε-factors at the real places via the Langlands parameter are the following (see [7, §4]):

$$L(s, \pi_\infty, \mathrm{spin}) = \Gamma_{\mathbf{C}}(s + \tfrac{c+\lambda_1+\lambda_2-1}{2})\Gamma_{\mathbf{C}}(s + \tfrac{c+\lambda_1-\lambda_2-1}{2}),$$
$$L(s, \pi_\infty, \mathrm{std}) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{C}}(s + \lambda_1 - 1)\Gamma_{\mathbf{C}}(s - \lambda_2),$$
$$\varepsilon(s, \pi_\infty, \mathrm{spin}, \psi_\infty) = (-1)^{\lambda_1},$$
$$\varepsilon(s, \pi_\infty, \mathrm{std}, \psi_\infty) = (-1)^{\lambda_1-\lambda_2}.$$

3. Whittaker functions on $\mathrm{GSp}(2,\mathbf{R})$

We recall the explicit formulas for Whittaker functions on G for the large discrete series representations. A nondegenerate unitary character ψ_{N_0} of N_0 is of the form $\psi_{N_0}(n(x_0,x_1,x_2,x_3)) = \exp\{2\pi\sqrt{-1}(c_0x_0+c_3x_3)\}$ with nonzero real numbers c_0 and c_3. We introduce the space

$$C^\infty(N_0\backslash G;\psi_{N_0}) := \{W \in \mathbf{C}^\infty(G,\mathbf{C}) \mid W(ng) = \psi_{N_0}(n)W(g),\ \forall(n,g) \in N_0 \times G\}$$

on which the group G acts by right translation. The restriction of a global Whittaker function to G is of moderate growth. Then we consider the subspace $C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi_{N_0}) := \{W \in C^\infty(N_0\backslash G;\psi_{N_0}) \mid W \text{ is of moderate growth}\}$ of $C^\infty(N_0\backslash G;\psi_{N_0})$. Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be the Lie algebra of G and G_0, respectively. Wallach's multiplicity one theorem asserts that for an irreducible $(\mathfrak{g},K)$-module π_∞, $\dim_{\mathbf{C}}\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi_{N_0})) \le 1$. If there is a nonzero intertwining operator $\Psi \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\infty, C^\infty_{\mathrm{mg}}(N_0\backslash G;\psi_{N_0}))$, then we call the image $W(v;*) = \Psi(v)$ of $v \in \pi_\infty$ the Whittaker function corresponding to $v \in \pi_\infty$.

Theorem 3.1. [8], [6] *Assume that* $1-\lambda_1 \le \lambda_2 \le 0$.

(i) We have

$$\dim_{\mathbf{C}}\mathrm{Hom}_{(\mathfrak{g}_0,K)}(D_{(\lambda_1,\lambda_2)}, C^\infty(N_0\backslash G_0;\psi_{N_0})) = \begin{cases} 1 & \text{if } c_3 > 0; \\ 0 & \text{if } c_3 < 0, \end{cases}$$

$$\dim_{\mathbf{C}}\mathrm{Hom}_{(\mathfrak{g}_0,K)}(D_{(-\lambda_2,-\lambda_1)}, C^\infty(N_0\backslash G_0;\psi_{N_0})) = \begin{cases} 0 & \text{if } c_3 > 0; \\ 1 & \text{if } c_3 < 0. \end{cases}$$

(ii) Let $c_3 < 0$. We take a standard basis $\{v_l^{(-\lambda_2,-\lambda_1)} \mid 0 \le l \le \lambda_1-\lambda_2\}$ of the minimal K_0-type $\tau_{(-\lambda_2,-\lambda_1)}$ of $D_{(-\lambda_2,-\lambda_1)}$. Then, up to a constant multiple, the radial part of Whittaker function corresponding to $v_l^{(-\lambda_2,-\lambda_1)}$ is given by

$$\begin{aligned}
&W(v_l^{(-\lambda_2,-\lambda_1)};\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})) \\
&= (2\sqrt{-1})^{-l}(|c_0|\frac{a_1}{a_2})^{\lambda_1-l+1}(|c_3|a_2^2)^{(\lambda_1+\lambda_2+1)/2}\exp(-2\pi|c_3|a_2^2) \\
&\times \frac{1}{(2\pi\sqrt{-1})^2}\int_{\sigma_1-\sqrt{-1}\infty}^{\sigma_1+\sqrt{-1}\infty}\int_{\sigma_2-\sqrt{-1}\infty}^{\sigma_2+\sqrt{-1}\infty}(\pi|c_0|\frac{a_1}{a_2})^{-2s_1}(4\pi|c_3|a_2^2)^{-s_2} \\
&\times (2s_1)_l\Gamma(s_1)\Gamma(s_1-s_2)\Gamma(s_2+1/2)\Gamma(s_2-\lambda_2+1/2)\,ds_1ds_2,
\end{aligned}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, and real numbers (σ_1,σ_2) are chosen such that $\sigma_1 > \sigma_2 > 0$.

4. Computation of archimedean zeta integrals

Using the explicit formulas for Whittaker functions, we can show the following.

Theorem 4.1. *Assume that $\pi_\infty \cong D_{(\lambda_1,\lambda_2)}[0]$ with $1-\lambda_1 \le \lambda_2 \le 0$. We take Whittaker function W_∞ for $D_{(\lambda_1,\lambda_2)}[0]$ and sections $f_{1,\infty}, f_{2,\infty}$ as follows:*

$$\begin{aligned}
&W_\infty(g) = W(v_{-\lambda_2}^{(-\lambda_2,-\lambda_1)};g), \quad g \in G, \\
&f_{1,\infty}(k) = 1, \quad k \in K_0, \\
&f_{2,\infty}(k) = \overline{\langle\tau_{(-\lambda_2,-\lambda_1)}(k)v_{-\lambda_2}^{(-\lambda_2,-\lambda_1)}, v_{-\lambda_2}^{(-\lambda_2,-\lambda_1)}\rangle}, \quad k \in K_0.
\end{aligned}$$

Then we have

$$Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty}) = c\cdot(\sqrt{-1})^{-\lambda_1}\frac{L(s_1,\pi_\infty,\mathrm{std})L(s_2,\pi_\infty,\mathrm{spin})}{\Gamma_{\mathbf{R}}(s_1+1)\Gamma_{\mathbf{R}}(2s_1)\Gamma_{\mathbf{R}}(2s_2+\lambda_1-\lambda_2+1)}.$$

Here $c = 2^{-2}(2\pi)^{-\lambda_2}\pi^{\lambda_1}\frac{\lambda_1!(-\lambda_2)!}{(\lambda_1-\lambda_2+1)!}$.

(Outline of proof) Recall that

$$Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty}) = \int_{\mathsf{Z}(\mathbf{R})\mathsf{N}_{12}(\mathbf{R})\backslash\mathsf{G}(\mathbf{R})} W_\infty(g)f_{1,\infty}(s_1,w_2g)f_{2,\infty}(s_2,w_1g)\,dg.$$

We denote by $\mathsf{Z}(\mathbf{R})\mathsf{N}_{12}(\mathbf{R})\backslash\mathsf{G}(\mathbf{R}) \ni g = n(x_0,0,0,0)n(0,0,0,x_3)\cdot\mathrm{diag}(ab,a,b^{-1},1)\cdot k$ ($x_0,x_3\in\mathbf{R}$, $a\in\mathbf{R}^\times$, $b>0$, $k\in K_0$) and consider the Iwasawa decomposition of w_ig. Then we have

$$Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty}) = \int_{\mathbf{R}_+^\times}\int_{\mathbf{R}^\times}\int_{K_0}\int_{\mathbf{R}^2} W_\infty(\mathrm{diag}(ab,a,b^{-1},1)k)$$

$$\times f_{1,\infty}\Big(s_1,u^{-1}\Big(\begin{pmatrix}1 & \\ & \frac{x_3-\sqrt{-1}a}{\sqrt{x_3^2+a^2}}\end{pmatrix}\Big)k\Big)f_{2,\infty}\Big(s_2,u^{-1}\Big(\frac{1}{\sqrt{x_0^2+b^2}}\begin{pmatrix}-x_0 & b\\ b & x_0\end{pmatrix}\Big)k\Big)$$

$$\times\Big(\frac{a^2b^2}{x_3^2+a^2}\Big)^{(s_1+1)/2}\Big(\frac{ab^2}{x_0^2+b^2}\Big)^{s_2+1/2}|a^3b^4|^{-1}\exp\{-2\pi\sqrt{-1}(x_0+x_3)\}dx_0dx_3dk\frac{da}{|a|}\frac{db}{b}.$$

We choose the data $(W_{1,\infty},f_{1,\infty},f_{2,\infty})$ as above and abbreviate $\tau=\tau_{(-\lambda_2,-\lambda_1)}$ and $v_l = v_l^{(-\lambda_2,-\lambda_1)}$. For $k\in K_0$, we have

$$W(v_{-\lambda_2};gk) = W(\tau(k)v_{-\lambda_2};g) = \sum_{0\le i\le\lambda_1-\lambda_2}\frac{\langle\tau(k)v_{-\lambda_2},v_i\rangle}{\langle v_i,v_i\rangle}W(v_i;g),$$

$$f_{2,\infty}(s_2,pk) = \sum_{0\le j\le\lambda_1-\lambda_2}\frac{\overline{\langle\tau(k)v_{-\lambda_2},v_j\rangle}\langle\tau(p)v_j,v_n\rangle}{\langle v_j,v_j\rangle}$$

where $p=p^{-1}=u^{-1}\big(\frac{1}{\sqrt{x_0^2+b^2}}\begin{pmatrix}-x_0 & b\\ b & x_0\end{pmatrix}\big)\in K_0$. Then Schur's orthogonal relation implies that

$$Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty})$$

$$=\frac{\lambda_1!(-\lambda_2)!}{(\lambda_1-\lambda_2+1)!}\sum_{0\le i\le\lambda_1-\lambda_2}\int_{(\mathbf{R}_+^\times)^2}\int_{\mathbf{R}^2}W_\infty(v_i;\mathrm{diag}(ab,a,b^{-1},1))\frac{\overline{\langle\tau(p)v_i,v_{-\lambda_2}\rangle}}{\langle v_i,v_i\rangle}$$

$$\times\Big(\frac{a^2b^2}{x_3^2+a^2}\Big)^{(s_1+1)/2}\Big(\frac{ab^2}{x_0^2+b^2}\Big)^{s_2+1/2}(a^3b^4)^{-1}\exp\{-2\pi\sqrt{-1}(x_0+x_3)\}dx_0dx_3\frac{da}{a}\frac{db}{b}.$$

We substitute the explicit formula for W_∞ and compute the integrations with respect to x_0,x_3,a,b:

$$Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty})$$

$$=\frac{c'}{(2\pi\sqrt{-1})^2}\int_{\tau_1-\sqrt{-1}\infty}^{\tau_1+\sqrt{-1}\infty}\int_{\tau_2-\sqrt{-1}\infty}^{\tau_2+\sqrt{-1}\infty}\frac{\Gamma(s_1+s_2-t_2+\frac{\lambda_1+\lambda_2-2}{2})\Gamma(s_2-t_2+\frac{\lambda_1+\lambda_2-2}{2})}{\Gamma(\frac{s_1}{2}+s_2-t_2+\frac{\lambda_1+\lambda_2-1}{2})\Gamma(\frac{s_1+1}{2})}$$

$$\times\sum_{\varepsilon\in\{0,1\}}\sum_{\substack{0\le i\le\lambda_1-\lambda_2\\ i\equiv\lambda_1+\varepsilon(\mathrm{mod}\ 2)}}(-1)^{m+(\lambda_1+\varepsilon-i)/2}(-\sqrt{-1})^\varepsilon\sum_{j,m}\binom{-\lambda_2}{j}\binom{\lambda_1}{i-j}\binom{j+(\lambda_1+\varepsilon-i)/2}{m}$$

$$\times \frac{\Gamma(\frac{s_1}{2}+s_2-t_1+\frac{2\lambda_1-\lambda_2-2m-i-1-\varepsilon}{2})\Gamma(\frac{s_1}{2}-t_1+\frac{\lambda_1-i-1+\varepsilon}{2})}{\Gamma(s_2+\frac{\lambda_1-\lambda_2-2m+1}{2})}$$
$$\times (2\sqrt{-1})^{-i}(2t_1)_i\Gamma(t_1)\Gamma(t_1-t_2)\Gamma(t_2+\frac{1}{2})\Gamma(t_2-\lambda_2+\frac{1}{2})\, dt_1dt_2.$$

Here $c' = \frac{\lambda_1!(-\lambda_2)!}{(\lambda_1-\lambda_2+1)!}2^{-(s_1+2s_2)-\lambda_1-\lambda_2-2}\pi^{-s_1-s_2-(\lambda_1+\lambda_2)/2+7/2}$. We use Barnes' first lemma for the integration over t_1, and collect the terms to find that

$$Z_\infty(s_1,s_2,W_\infty,f_{1,\infty},f_{2,\infty})$$
$$=\frac{\Gamma(\frac{s_1+\lambda_1}{2})\Gamma(\frac{s_1+\lambda_1-1}{2})}{\Gamma(\frac{s_1+1}{2})\Gamma(s_2+\frac{\lambda_1-\lambda_2+1}{2})}$$
$$\times\frac{c'}{(2\pi\sqrt{-1})}\int_{\tau_2-\sqrt{-1}\infty}^{\tau_2+\sqrt{-1}\infty}\Gamma(s_2-t_2+\frac{\lambda_1+\lambda_2-2}{2})\Gamma(t_2-\lambda_2+\frac{1}{2})\Gamma(t_2+\frac{1}{2})$$
$$\times\begin{cases}\displaystyle\sum_{q=0}^{-\lambda_2/2}\frac{(\sqrt{-1})^{-\lambda_1+\lambda_2+2q}2^{2q+\lambda_2}(-\lambda_2)!}{(2q)!(-\frac{\lambda_2}{2}-q)!}\Gamma(\frac{s_1}{2}-t_2+\frac{\lambda_2+1}{2}+q)\,dt_2 & \text{if } -\lambda_2 \text{ is even;}\\ \displaystyle\sum_{q=0}^{(1-\lambda_2)/2}\frac{(\sqrt{-1})^{1-\lambda_1+\lambda_2+2q}2^{2q+\lambda_2+1}(-\lambda_2)!}{(2q+1)!(\frac{-\lambda_2-1}{2}-q)!}\Gamma(\frac{s_1}{2}-t_2+\frac{\lambda_2}{2}+q)\,dt_2 & \text{if } -\lambda_2 \text{ is odd.}\end{cases}$$

Using Barnes' first lemma for $\int_{t_2}$ again, and taking the summation over q, we can get the assertion. □

Remark 1. When π_∞ is isomorphic to the class one principal series, coincidence of archimedean zeta integral and the product of local L-factors (divided by normalizing factors of two Eisenstein series) is shown ([3]).

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